Sensorimotor Analogies in Learning Abstract Skills and Knowledge: Modeling Analogy-Supported Education in Mathematics and Physics

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Abstract

In this summary report I give an account of research conducted over the last two years, showing the suitability and the advantages of applying computational analogy-engines in the analysis and design of analogy-based methods and tools in teaching and education. This overview constitutes the conclusion of the first phase of a multi-stage effort trying to introduce computational models of analogy also to education and the learning sciences, thus opening up these fields to computational tools and methods not only on an instrumental level, but also in analytical, conceptual, and design-oriented studies. I locate the "analogy-engines in the classroom" research program within the bigger schemes of studying human creativity and computational creativity, provide an introduction to the theoretical underpinnings of the endeavor, and revisit three worked out case studies serving as proofs of the feasibility of the overall approach.

1 Big Small Creative Leaps During Developmental Learning and Education

Creativity research very often tends to focus on one of two directions: everyday creativity, i.e., the form of creative activity occurring in nearly all people, and the creativity of the genius, reserved for the great creative minds of each generation. In scientific terms, both directions have received their respective theories and denominations, with the day-to-day class of creative acts normally being referred to as "little-c" creative, and the stroke-of-genius category going by the name of "Big-C" creative (cf., e.g., (Kaufman and Sternberg 2006)).[1] But as appealing and intuitive this simple categorization may be, over the last years more and more researchers realized that the bimodal view of creativity leaves out at least two crucial forms of creativity (Kaufman and Beghetto 2009): "mini-c" creativity, creativity inherent in learning and personal discovery processes (as, e.g., all the time happening during children's development and education), and "Pro-c" creativity, the developmental and effortful progression beyond little-c that represents professional-level expertise in any creative area.

In this report, I want to focus on mini-c creativity, and more specifically on analogy-related aspects of creative leaps during learning and development in the first years of school. Mini-c creativity fundamentally differs from the existing accounts of everyday (little-c) or eminent (Big-C) creativity as it targetedly refers to the creative process involved in the construction of personal knowledge and understanding. (Beghetto and Kaufman 2007) define the mini-c dimension of creativity as follows: "*the novel and personally meaningful interpretation of experiences, actions, and events*". This makes mini-c creativity an exclusively intrapersonal notion — novelty and meaningfulness of the newly acquired interpretations and insights only have to be perceived on a personal level, but do not necessarily have to be acknowledged by anybody else.

And, indeed, this is what happens during a child's discovery of the world and during almost all the educational career from preschool to A levels (and maybe even throughout university). Although not perceived as a significant creative act, and seemingly just natural and commonsensical to most bystanders, the first time a child masters simple mathematical concepts such as, for instance, addition, subtraction, or multiplication, or understands something about the nature of not directly observable physical phenomena such as, for example, the flow of electric current, on an intrapersonal level these insights constitute major breakthroughs in the subjective perception of the child — and also open up new ways to conceptualize, understand, and subsequently utilize its environment and the world.

One of the main tasks of our educational system(s) is to guide children throughout their developmental trajectory and to facilitate and (partially) trigger the occurrence of these mini-c moments: Whilst some insights might also be obtained without any kind of teaching, it is undoubted that only education allows the (mostly) effective and efficient acquisition of knowledge and skills observable throughout the years a child spends in the different levels of the school system, carrying it from basic counting to differential equations and probability distributions, and from simple naive physics to thermodynamics and Einsteinian relativity.

But education cannot be reduced merely to the presentation of facts and knowledge, also the way in which teach-

[1] A similar categorization has been presented in (Boden 2004). Boden distinguishes acts of p-creativity (personal; similar to the little-c category) from moments of h-creativity (historical; the couterpart of Big-C creative moments).

ing happens plays a crucial role: Different approaches and philosophies seem more or less likely to actually trigger genuine insight processes which allow the learners to eventually ascend to a new level of understanding (instead of simply trying to memorize the given data without any abstraction or generalized understanding). Over the last years significant effort has been put into research trying to identify the most suitable means of insight- and discovery-based teaching — with analogy-based approaches to education scoring high in almost all respects.

2 Analogy in Education

Analogical reasoning (i.e., the ability of perceiving and operating on dissimilar domains as similar with respect to certain aspects based on shared commonalities in relational structure or appearance) is considered essential for learning abstract concepts (Gentner, Holyoak, and Kokinov 2001) or procedures (Ross 1987), for transferring representations across contexts (Novick 1988) or for adapting existing knowledge to newly encountered scenarios and contexts (Holyoak and Thagard 1995), and in general for children's process of learning about the world (Goswami 2001).

Concerning an educational context, analogies facilitate learners' construction processes of new ideas and conceptions on the grounds of already available concepts (Duit 1991), and can foster understanding by abstracting the important ideas from the mass of new information, helping to clarify the system boundaries and dynamics, and providing a language for framing a scientific explanation (Arnold and Millar 1996). Also, analogy can be used for facilitating the understanding of concepts and procedures in abstract and formal domains such as mathematics, physics or science (Richland, Zur, and Holyoak 2007; Guerra-Ramos 2011). On the downside, there is evidence indicating that unsuccessful analogies may produce misunderstandings and can result in harmful misconceptions (Clement 1993; Zook and Maier 1994). But notwithstanding these results (and the corresponding studies) on analogy in a teaching context, unfortunately the current general level of knowledge about analogy as an instructional device in everyday practice still has to be considered quite low (Akgul 2006) and, given its importance, urgently needs further research.

2.1 Sensorimotor Experiences Supporting Analogy-Based Discovery and Learning

Over the last two decades, a growing number of mathematical education researchers have emphasized the embodied nature of mathematical understanding, assigning sensorimotor action a crucial role in mathematical activity. One of the major struggles in this line of research has been to explain how formal, mathematical ideas can emerge from concrete sensorimotor experiences, and how existing abstract knowledge can be interlinked with embodied aspects in order to allow for further evolution and development of already acquired mathematical insights.

A candidate solution to this problem has been found in a phenomenon intrinsically related to analogy[2], namely in the notion of embodied metaphors, i.e., metaphors whose source is bodily experience. (Sfard 1994) suggested that these types of metaphor could possibly play the role of translating between bodily experiences into the less concrete realm of mathematical ideas, thereby emphasizing the role of metaphor in creating the universe of abstract ideas. But the real breakthrough for the metaphor-based view happened with the publication of (Lakoff and Núñez 2000), in which the authors try to systematically explain how this process works in mathematics and how a handful of basic metaphors could possibly give rise to our conception of mathematics based on three fairly grounded assumptions: that everything humans know must be derived in some way from sensorimotor experiences; that essential aspects of human thinking are not accessible to awareness; and, that humans understand through conceptual metaphor. And whilst several of the derived claims still remain controversial or did not withstand closer empirical evaluation, the authors' account served as starting point for several interesting proposals advancing the understanding of the interaction between sensorimotor experiences and the acquisition of abstract skills and knowledge beyond the state of the art at the time (cf., e.g., (Schiralli and Sinclair 2003) for an insightful reply to and further development of the ideas presented in the original book).

Embodied notions of mathematical understanding very often conceptually tie into the "functional thinking" approach to mathematics (cf., e.g., (Schwank, Gelfman, and Nardi 1999)). As opposed to a "predicative thinking"-style understanding of relationships within mathematics, which uses equality as ordering principle when conceptualizing mathematical structures (i.e., mathematics being conceptualized on the basis of the repeated applicability of certain predicates), the "functional thinking" perspective bases its conceptualizations on differences between mathematical concepts which can then be used to conceive of a construction process for the respective class of structures (i.e., mathematics being conceptualized on the basis of repeated constructive steps) and, thus, offers itself to a more sensorimotor grounded approach.

Exploiting this relationship between embodied metaphors and functional thinking permits the construction of teaching tools specifically targeting the embodied, functional aspects of mathematical understanding by allowing learners to discover new mathematical insights and concepts via free sensorimotor interaction and play. Whilst still in a fairly early stage as a field of study, researchers in this line of activity already have developed and field-tested several prototypical setups such as, for instance, the "Calculation Circular Staircase" or the "Number Highrise" (cf. Fig.1 and Sect. 5).

And clearly, sensorimotor approaches to teaching and learning abstract skills and knowledge are not limited to mathematics but can also be applied in educational contexts in other formal sciences such as, for instance, physics. Many notions in physics allow for the use of models for exemplification and better illustration, with the models in turn often allowing for the integration of active sensorimotor involve-

[2] I here take a stance conceptually unifying metaphor with processes of analogy and similarity similar to the approach presented in (Gentner, Falkenhainer, and Skorstad 1988; Gentner et al. 2001).

Figure 1: From top left, in counterclockwise order: The small and the big "Calculation Circular Staircase" (Schwank, Aring, and Blocksdorf 2005), and the "Number Highrise" together with a toy figure used when playing with the highrise (Schnalle and Schwank 2006).

ment of the learners. One example for such a model is the string circuit analogy for gaining a basic understanding of electric current (Asoko 1996; Guerra-Ramos 2011) used in science classes for 8 to 9 year old children: On the embodied side, the analogy uses a representation which places participants in a circle, making them loosely support with their hands a continuous string loop that one person subsequently makes circulate (cf. Sect. 5).

3 Educational Analogies and AI

But analogy is not only an object of study in psychology and the learning sciences, but since the very beginnings has also been actively investigated in artificial intelligence (AI), bringing forth numerous computational frameworks and systems for automated analogy-making and analogical reasoning. And, indeed, computational analogy frameworks also found entrance into the context of education and teaching:

- In (Thagard, Cohen, and Holyoak 1989), the authors present a theory and implementation of analogical mapping that applies to explanations of unfamiliar phenomena as e.g. used by chemistry teachers (explanations providing systematic clarification as well as explanations giving a causal account of why something happened).
- (Forbus et al. 1997) show how an information-level model of analogical inferences (supporting reasoning about correspondences and mappings), together with techniques for the structural evaluation of analogical inferences, can be incorporated in a case-based coach that is being added to an intelligent learning environment, whilst (Lulis, Evens, and Michael 2004) analyze the use of analogies in human tutoring sessions for medical students in order to build an interactive electronic tutoring system capable of also applying analogy.
- (Siegler 1989) conjectures how the Structure-Mapping Engine (Falkenhainer, Forbus, and Gentner 1989) as prototypical analogy-engine could be used to gain insights about developmental aspects of analogy use.

4 Analogy-Engines in the Classroom

Building on the outcome of these and similar research efforts, starting out with (Besold 2013), I advocate to expand research applying analogy-engines to problems from teaching and learning into a proper program in its own right, opening up a new application domain to computational analogy-making and helping to shed light on the role of analogy in mini-c creative acts during child development and eduction.

Since the beginning of these efforts, three possible classes of tasks have been identified as main scenarios for the beneficial use of computational analogy systems in an educational context:

- **Modeling and analysis:** Symbol-based analogy-engines can be put to use for modeling and understanding the conceptual mode of operation of analogies in a teaching context, addressing a level of detail situated between what (Marr 1982) considered the level of computational theory and the level of representation and algorithm: Although there will most likely not be a perfect match on algorithmic level between the mental processes triggered in the student's mind and the inner workings of the analogy model, the correspondence can be deeper rooted than residing exclusively on the computational level, not only covering input/output behavior in a purely functionalist way but also addressing basic properties and principles of the mechanisms at work. Also, as proposed by (Siegler 1989), developmental aspects of analogy use may be addressed by taking into account results about children's cognitive development and cognitive capacities, accordingly constraining the mechanisms used by the analogy engine.
- **Exploration and testing:** Heuristics-based analogy-making frameworks, or frameworks allowing for an incorporation of heuristics-like aspects to the guiding and steering mechanisms of the engine, may be used for exploring, developing and testing analogies for a teaching context. Provided that, the heuristics applied by the computational system are adapted as to mirror children's cognitive capacities and limitations in adequate ways, the analogy-engine can be used for testing the suitability of analogies considered for the use in a specific teaching situation. Also, in an exploration-type mode, the system could be put to use in discovering possible analogies between two domains that are given by the teacher for application in a specific educational context, thus simulating children's

analogy generation and understanding in the respective situation.

- **Discovery and guidance:** Given two domains by the teacher, the analogy-engine can be used for discovering what analogies possibly arise between these domains and how the analogy-making process might have to be guided (e.g., what framing facts have to be included in the initial domain theories) to obtain one specific, previously planned analogy as result of the process. Assuming similarity between the mechanisms implemented in the analogy-making system and the principles and constraints applying to analogy-making in pupils, the guiding constraints obtained during the run of the system may then be used for re-designing the initial domain theories.

5 Three Worked Examples

In order to provide factual grounding and initial worked examples for the just proposed applications of computational analogy-engines, (Besold 2013), (Besold, Pease, and Schmidt 2013), and (Besold and Kühnberger 2014) feature three case studies (mostly addressing the "modeling and analysis" and the "exploration and testing" scenarios). In all cases, the Heuristic-Driven Theory Projection (HDTP) analogy-making framework (Schmidt et al. 2014) (i.e., a logic-based, mathematically sound framework computing analogical relations and inferences between domain formalizations) was applied to modeling real-world examples taken from a classroom context. HDTP had been chosen as modeling framework due to its symbolic nature and the explicit computation of a generalized theory between the source and the target domain of the respective analogy (the explicit representation of domain elements and the resulting explicated conceptual structure of the analogies allows for a direct evaluation against the background of theories and results from the learning sciences and experimental classroom data).

5.1 The String Circuit Analogy

(Besold 2013) provides an HDTP model of the aforementioned string circuit analogy for gaining a basic understanding of electric current used in science classes for 8 to 9 year old children. As described, participants are placed in a circle, loosely supporting with their hands a continuous string loop that one person subsequently makes circulate. This analogy builds on basic prior insights acquired when playing with simple electric circuits, such as the recognition that "energy", "electricity" or "power" seems to leave a battery, and motivates students to conjecture about how their observations from the string circuit's workings might translate to the non-observable processes and mechanisms within the electric circuit, emphasizing some crucial main ideas as, for example, the provision of energy by the battery, and that energy is carried by current. The HDTP model supported findings previously published by (Guerra-Ramos 2011) concerning the alignment of domain elements between the string circuit and the electric circuit domains (cf. Fig. 2) and also provided a more detailed account of the necessary conceptual transfers and required prior theory elements for each domain.

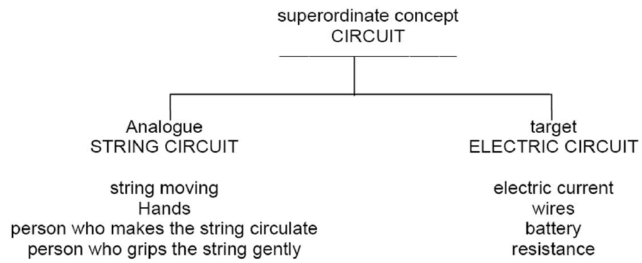

Figure 2: Representation of the "string circuit" analogy as given in (Guerra-Ramos 2011).

5.2 The Calculation Circular Staircase

(Besold, Pease, and Schmidt 2013) give a detailed and fairly complex formal model of the analogy-based Calculation Circular Staircase (Schwank, Aring, and Blocksdorf 2005), applied in teaching basic arithmetics and the conception of the naturals as ordinal numbers to children attending their initial mathematics classes in primary school. The Calculation Circular Staircase, (i.e., a teaching tool shaped like a circular staircase with the steps being made up by incrementally increasing stacks of balls, grouped in expanding circles of ten stacks per circle corresponding to the decimal ordering over the naturals; cf. Fig. 1) offers children a means of developing an understanding of the interpretation of numbers as results of transformation operations by enabling a mental functional motor skill-based way of accessing the foundational construction principles of the number space and the corresponding basic arithmetic operations when playing with the tool. The HDTP model now gives a precise account of how the structure of the staircase and the declarative counting procedure memorized by children in school interact in bringing forth the targeted conception of the natural number space (and thus also nicely ties in e.g. with the recent cognitive science discourse on numerical concept learning (Piantadosi, Tenenbaum, and Goodman 2012)). By providing a detailed formal description of the involved domains, also sketching how the domains relate to each other in terms of their joint generalization and how this relation can be used to transfer knowledge from the staircase domain into the number domain, the structural relations and governing laws underlying the Calculation Circular Staircase as teaching model of the natural number domain are explicated, and it is shown how the identified constructive and transformation-based conceptualizations then also can provide additional support and a deeper-rooted model for the children's initially very flat and sparse conception of the number domain.

5.3 The Number Highrise

In (Besold and Kühnberger 2014) a similar worked example has been built for the Number Highrise teaching tool for understanding multiplicative relations over the natural numbers (Schnalle and Schwank 2006). The Number Highrise is a mathematical toy world designed for exploring the space of the natural numbers up to 100 making explicit use of multiplication-based relations. It consists of 10 so called hallways each made up by 100 pearls (cf. Fig. 1). In each hallway the pearls are connected in chunks going in accordance with one of the basic times tables for numbers between 1 and 10. The children interact with toy figures which can travel the different hallways up and down. If two or several toy figures within their respective hallway can reach platforms on equal height with each other, they are called neighbors and can celebrate a party together. In class the Number Highrise can be used in several ways: Documented uses range from teaching the basic conception of the natural number space, through the discovery of concepts like the least common multiple and the greatest common factor, to finding and exploring the notion of prime numbers. As it was the case with the Calculation Circular Staircase, the HDTP model explicates the structural relations and governing laws underlying the Number Highrise as teaching model: Amongst others, it becomes clear how the constructive relations and insights obtained when playing with the toy world can give meaning to the multiplication relation via the assignment of the constructive process of climbing the highrise and the corresponding platforms for the intermediate steps and the result, and it can be reconstructed how analogical transfer from the Number Highrise into the children's conception of the number space, building upon the already pre-existing basic structure of natural numbers and times tables, can trigger the acquisition of the more complex idea of the least common multiple.

Summary: In all three case studies the respective formal model shows to be highly useful in uncovering the underlying structure of the method or teaching tool, together with the consecutive steps of reasoning happening at an intermediate level between computation and algorithm. This constitutes a valuable addition to the usual means of analysis by observation and (often) qualitative or (less frequently) quantitative analysis over groups of learners applied in educational studies and the learning sciences. Also, by providing three different examples taken from two different domains and using two different modes of integrating sensorimotor experiences into the learning process (i.e., an interactive paradigm with specifically designed tools in the case of the Calculation Circular Staircase and the Number Highrise, and a paradigm incorporating the learner herself into the analogical model in the string circuit analogy) a certain degree of domain and mode generality and universal usability of the proposed approach and methods has been shown.

6 Future Work and Conclusion

By providing detailed formal descriptions of the involved domains and their relation in terms of their joint generalization and the corresponding possibility for knowledge transfer, the computational analogy models explicate the structural relations and governing laws underlying the respective teaching tools and methods. Also they show how the identified constructive and transformation-based conceptualizations then can provide support and a deeper-rooted model for the children's initially very flat and sparse conceptions of the corresponding domains of abstract skills and knowledge — and thus shed light on crucial steps in the mini-c creative act of learning and discovery.

The mentioned models of the string circuit analogy, the Calculation Circular Staircase, and the Number Highrise have been built mostly using data from qualitative analysis and observational studies conducted by education researchers. Now that the models have been established in the first phase of the overall "analogy engines in the classroom" program, a natural next step for a second phase will therefore be to establish contact with psychological evidence, trying to address some of the following questions: Are the formalization and the modeling assumptions realistic when compared to psychological data? Given what parts of the learning process are hard and easy for children, are those things hard or easy for the system? Are the misconceptions that the system exhibits similar to the misconceptions that can be found in young learners (and vice versa)?

In general, modeling educational analogies sheds new light on a particular analogy, in terms of which information is transferred, what the limitations of the analogy are, or whether it makes unhelpful mappings; and what potential extensions might be needed. On this basis, I hope to acquire a deeper understanding of the basic principles and mechanisms underlying analogy-based learning in fairly high-level and abstract domains (and, thus, in turn insights into the process underlying mini-c creative acts). This does not only offer new chances for artificial intelligence (AI) and computational creativity (CC) outsides their original territory by bridging the disciplinary gap to researchers in education, learning sciences, and developmental cognitive science, but — presupposing that one way for AI and CC to succeed is to understand and subsequently re-implement human mental processes — also is highly relevant within AI and CC themselves, resulting in a dual-use scenario for the resulting models. By making the individual, successive processing stages accessible on a formal level, the models allow for an analysis and possible re-implementation of the analogy-based reasoning in an artificial system, thus re-using the insights into the developmental process in children for the purpose of AI and CC. In doing so, two of the core questions in the (re-) creation of intelligence and creativity are being addressed: the acquisition and formation of concepts, and the transfer of concepts and reasoning schemas across domains.

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